MICRO-428: Metrology

Week Three: Elements of Statistics

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Exercise 1: Memorylessness

- Demonstrate that the exponential distribution is memoryless.

Exercise 2: Stationary Processes

- Let α and ω be two known constants and β a uniform RV with PDF:

$$f_U(\beta) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \beta \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

- Let $X(t)$ be the RP:

$$X(t) = \alpha \cos(\omega t + \beta)$$

- Demonstrate that the RP $X(t)$ is Wide Sense stationary and, eventually, ergodic.

Exercise 3: Estimation using MLE

- MLE: Given a sample of n independent experiments $x_1, x_2, \dots, x_n$, and defined θ the parameters of the RV, we define the likelihood as:

$$L(x_1, x_2, \dots, x_n; \theta) = \prod_{i=1}^{n} f_X(x_i, \theta)$$

- The MLE estimator is a value $\hat{\theta}$ such that L is maximized.

- A perfect single-photon detector (efficiency of 100%, no jitter, no DCR, etc..) is working in ultra-low photon rate regime (e.g. as a exoplanet space telescope) connected with a TDC (time-to-digital converter). The detector collects, in 50 ms, 4 photons ($t_{arrival}$ = 1, 20, 35, 38 ms). Estimate $\hat{\lambda}$.

Homework 1: Estimation using MLE

- The measured fluorescence lifetime curve is modeled as the convolution between the exponential function and the instrument response function (IRF). Considering that we measure fluorescence lifetime with an ideal setup (IRF is the Dirac function), calculate the maximum likelihood estimator of fluorescence lifetime given the arrival time of N photons.

Homework 2: Random Walk

- The random walk is a random process which can be used to model the path resulting from random steps to the left ($X_j = 1$) or to the right ($X_j = -1$) starting from the position 0.
- Show that the variance of the random walk is maximum for

$$P\{X_j = 1\} = p = 0.5$$

HINT: the final position of N-step random walk is the sum of X_j :

$$Y = \sum_{j=1}^{N} X_j$$